CORRELATION INEQUALITIES FOR SPIN GLASS IN ONE DIMENSION

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**Abstract**

We prove two inequalities for the direct and truncated correlation for the nearest-neighboor one-dimensional Edwards-Anderson model with symmetric quenched disorder. The second inequality has the opposite sign of the GKS inequality of type II. In the non symmetric case with positive average we show that while the direct correlation keeps its sign the truncated one changes sign when crossing a suitable line in the parameter space. That line separates the regions satisfying the GKS second inequality and the one proved here.

# 1 Introduction and Results

In a recent paper [CL] a correlation inequality was proved for spin systems with quenched symmetric random interaction in arbitrary dimension, extending a previous result for the Gaussian case [CG]. That inequality yield results for spin glasses similar to those obtained for ferromagnetic systems from the first GKS inequality [Gr, Gr2, KS] e.g. it gives monotonicity of the pressure in the volume and bounds on the surface pressure. Other inequalities were considered: in particular the extension to non-symmetric interactions and possible versions of a second type GKS inequality.

In this work we study the $d = 1$ case with nearest neighboor interaction. In the same spirit of the GKS systems no assumption of translation invariance is made on the

interaction distributions and by consequence our results cannot be obtained by an exact solution. We prove that both the inequality of the first type does extend to the non symmetric case and that an inequality of the second type holds indeed in the symmetric case. A similar result with a complete proof of inequalities of type I and II has been obtained so far only in the Nishimori line [CMN, MNC].

Let us consider a chain with periodic boundary condition

$$H(\sigma, J) \;=\; -\sum_{i=1}^{N} J_i \sigma_i \sigma_{i+1}$$

with $\sigma_{N+1} = \sigma_1$. The random variables $J_i$ have independent distributions $p(J_i)$. Those fulfills one of the three following hypothesis, which will be called system I, II and III in the remaining part of the paper:

**I)**

$$p(|J_i|) \geq p(-|J_i|), \quad \forall i \;\; and \;\; \forall |J_i| \in \mathbb{R}^+$$

**II**) the $J_i$ are symmetric around a positive mean $\mu_i > 0$:

$$p(\mu_i + |J_i|) = p(\mu_i - |J_i|), \quad \forall i \;\; and \;\; \forall |J_i| \in \mathbb{R}^+$$

In the case of discrete variables: $J_i = \mu_i \pm J^{(i)}$, $p(\mu_i + J^{(i)}) = p(\mu_i - J^{(i)}) = 1/2$, we assume that $J^{(i)} > \mu_i$ (see below for further explanations) and we introduce the notations:

$$a_i = \mu_i + J^{(i)}$$
$$-b_i = \mu_i - J^{(i)}$$
$$a_i, b_i > 0$$

**III**) the $J_i$ are discrete variables taking on values $\pm J^{(i)}$ with $J^{(i)} > 0$ such that:

$$where \quad p_i = p(J^{(i)}), \quad q_i = p(-J^{(i)}) \;,$$

and

$$\alpha := \prod_i (p_i - q_i) \; \geq 0 \;.$$

Let $\omega_h$ the thermal average of the quantity $\sigma_h\sigma_{h+1}$, $\omega_{h,k}$ that of the quantity $\sigma_h\sigma_{h+1}\sigma_k\sigma_{k+1}$ and $\mathrm{Av}\,[\cdot]$ the average over the quenched disorder.

Our main results are:

**Proposition 1.1** *For all three systems:*

$$\mathrm{Av}\,[J_h\omega_h] > 0, \qquad \forall h = 1...N \tag{1.1}$$

**Proposition 1.2** *For systems I and III with $\alpha = 0$:*

$$\mathrm{Av}\,[J_hJ_k(\omega_{hk} - \omega_h\omega_k)] < 0, \qquad \forall h,k = 1...N, \quad h \neq k \tag{1.2}$$

**Proposition 1.3** *For system III, with $\alpha > 0$, the following properties hold:*

*$\forall l$, there exists in the $(J^{(l)},\alpha)$ quadrant, a curve $\alpha(J^{(l)})$ such that the quantity*

$$\mathrm{Av}\,[J_hJ_k(\omega_{hk} - \omega_h\omega_k)] \tag{1.3}$$

*changes its sign from negative to positive when crossing the curve $\alpha(J^{(l)})$ by increasing $\alpha$ and such that on the curve $\alpha(J^{(l)})$*

$$\mathrm{Av}\,[J_hJ_k(\omega_{hk} - \omega_h\omega_k)] = 0, \qquad \forall h,k = 1...N, \quad h \neq k\ . \tag{1.4}$$

*Moreover $\mathrm{Av}\,[J_hJ_k(\omega_{hk} - \omega_h\omega_k)]$ is increasing in $\alpha$ along the $J^{(l)} = const$ lines.*

# 2 Proofs

We start by proving the following lemmata.

**Lemma 2.1** *System III can be rewritten as:*

$$H(\tau, K) = -K_N\tau_N\tau_{N-1} - \sum_{i=1}^{N-1} J^{(i)}\tau_i\tau_{i+1} \tag{2.5}$$

*with:*

$$K_N = J^{(N)}\prod_{i=1}^{N}\mathrm{sgn}(J_i) = \pm J^{(N)}\ . \tag{2.6}$$

*Setting* $P = \text{prob}(K_N = J^{(N)})$ *and* $Q = \text{prob}(K_N = -J^{(N)})$ *we have:*

$$\begin{aligned} P &= \frac{1 + \prod_i (p_i - q_i)}{2} && (2.7) \\ Q &= \frac{1 - \prod_i (p_i - q_i)}{2} && (2.8) \end{aligned}$$

**Lemma 2.2** *Consider system II with discrete variables and assume that* $\mu_h = 0$ *for at least one* $h$*. Such a system can be rewritten as:*

$$H(\tau, K) = -\sum_{i=1}^{N} K_i \tau_i \tau_{i+1}$$

*where:*

$$\begin{aligned} K_h = J_h &= \pm a_h, \quad a_h > 0 && (2.9) \\ K_i &= \begin{cases} a_i > 0 \\ b_i > 0 \end{cases} && (2.10) \end{aligned}$$

*the two cases having probability* $1/2$*.*

**Proof of Lemma 2.1**

The proof is based on the Gauge transformation $\alpha_j = \prod_{1 \le i < j} \text{sgn}(J_i)$, for $2 \le j \le N$ $\alpha_1 = 1$. Set $\tau_i = \alpha_i \sigma_i$ $H$ is given by (2.5) with $K_N = J^{(N)} \prod_{i=1}^{N} \text{sgn}(J_i)$. We have now to compute the new probability measure for $\prod_{i=1}^{N} \text{sgn}(J_i)$. Clearly the expectation

$$\text{Av}\left[\prod_{i=1}^{N} \text{sgn}(J_i)\right] = \prod_{i=1}^{N} \text{Av}\left[\text{sgn}(J_i)\right] = \prod_{i=1}^{N} (p_i - q_i) = P - Q \; .$$

**Proof of Lemma 2.2**

Group the bond configurations in couples that only differ for the sign of $J_h$ and Gauge transform them using the same transformation of Lemma 3.1 What we obtain is:

$$\begin{aligned} K_i &= |J_i| > 0 \\ K_h &= J_h = \pm J^{(h)} \end{aligned}$$

Moreover, since $p(K^{(h)}) = p(J^{(h)})$ or $p(K^{(h)}) = p(-J^{(h)})$, $p(K^{(l)}) = 1/2$ for all $l$.

Introduce, for system **III**, the following shorthand notations:

$$C_i := \cosh(K^{(i)}) = \cosh(J^{(i)}); \qquad S_i := \sinh(K^{(i)}) = \sinh(J^{(i)})$$

**Proof of Proposition 1.1**

First we prove the thesis for discrete variables (system II and III). The partition function and correlation of an $N$ spins chain with periodic boundary conditions can be written as:

$$Z = \prod_i C_i + \prod_i S_i \tag{2.11}$$

$$\omega_h = \frac{1}{Z}\left[S_h \prod_{i\neq h} C_i + C_h \prod_{i\neq h} S_i\right] \tag{2.12}$$

**System III** Using Lemma 2.1 one has:

$$\begin{aligned} \mathrm{Av}_{\{J\}}[J_h\omega_h] &= \mathrm{Av}_{(K_h)}[K_h\omega_h] = K^{(h)}\left\{P\omega|_{k_h=k^{(h)}} - Q\omega|_{k_h=-k^{(h)}}\right\} = \\ &= K^{(h)}\left\{Q\left[\omega|_{k_h=k^{(h)}} - \omega|_{k_h=-k^{(h)}}\right] + (P-Q)\omega|_{k_h=k^{(h)}}\right\} \geq 0 \end{aligned}$$

due to the first Griffith's inequality for ferromagnetic systems.

**System II** (discrete variables) Since the pressure is a convex function of the $\mu_i$'s we can prove our theorem for $\mu_h = 0$. If for some $i$ $J^{(i)} \leq \mu_i$ the variable $J_i$ takes positives values and it doesnt influence the sign of the average. Now, using lemma 2.2 and observing that $P = Q = 1/2$ and the $J$ average is a linear combination of $K_h$ average with all positive remaining $K_i$, we have the thesis with the same steps as before. The extension to the continuos case is obtained by the usual method of integrating over the positive parts of the $J_i$ distributions.

**Proof of Proposition 1.2**

For discrete variables (system III), using the standard hyperbolic expansion:

$$\omega_{hk} = \frac{1}{Z}\left[S_h S_k \prod_{i\neq h,k} C_i + C_h C_k \prod_{i\neq h,k} S_i\right] \Rightarrow$$

$$
\begin{aligned}
\omega_{hk}-\omega_h\omega_k &= \frac{1}{Z^2}\left\{\left(S_hS_k\prod_{i\neq h,k}C_i + C_hC_k\prod_{i\neq h,k}S_i\right)\left(\prod_i C_i+\prod_i S_i\right)+\right.\\
&\qquad \left.-\left(S_h\prod_{i\neq h}C_i + C_h\prod_{i\neq h}S_i\right)\left(S_k\prod_{i\neq k}C_i + C_k\prod_{i\neq k}S_i\right)\right\} =\\
&= \frac{1}{Z^2}\left\{\prod_{i\neq h,k}(C_iS_i)\cdot\left(C_h^2C_k^2+S_h^2S_k^2-C_h^2S_k^2-S_h^2C_k^2\right)\right\} =\\
&= \frac{\displaystyle\prod_{i\neq h,k}C_iS_i}{\left(\displaystyle\prod_i C_i+\prod_i S_i\right)^2}
\end{aligned}
$$

If at least one of the random variables is symmetric we have: $P = Q = 1/2$; using lemma 2.1 one has:

$$
\begin{aligned}
\mathrm{Av}[K_hK_k(\omega_{hk}-\omega_h\omega_k)] &= J^{(k)}\cdot \mathrm{Av}_{(K_N)}\left[\frac{K_h\cdot\displaystyle\prod_{i\neq h,k}(C_iS_i)}{(\prod_i C_i+\prod_i S_i)^2}\right] =\\
&= J^{(k)}J^{(h)}\prod_{i\neq h,k}(C_iS_i)\cdot\frac{1}{2}\left\{\frac{1}{(\prod_i C_i+\prod_i S_i)^2}-\frac{1}{(\prod_i C_i-\prod_i S_i)^2}\right\} =\\
&= -2\,J^{(k)}J^{(h)}\prod_{i\neq h,k}(C_iS_i)\cdot\frac{\prod_i(C_iS_i)}{(\prod_i C_i^2-\prod_i S_i^2)^2} < 0
\end{aligned}
$$

The extension to the continuous case is as above.

**Proof of Proposition 1.3**

Let $\alpha > 0$ or equivalently $P = \frac{1+\alpha}{2} > \frac{1}{2} > Q = \frac{1-\alpha}{2}$.

We obtain analogously as before:

$$
\mathrm{Av}[K_hK_k(\omega_{hk}-\omega_h\omega_k)] = J^{(k)}J^{(h)}\prod_{i\neq h,k}(C_iS_i)\frac{P(\prod C_i-\prod S_i)^2-Q(\prod C_i+\prod S_i)^2}{(\prod C_i^2-\prod S_i^2)^2} =
$$

$$
= \frac{J^{(k)}J^{(h)}\prod_{i\neq h,k}(C_iS_i)}{(\prod_i C_i^2-\prod_i S_i^2)^2}\cdot\{(P-Q)(\textstyle\prod_i C_i^2+\prod_i S_i^2)-2\prod_i(C_iS_i)\}
$$

The sign of the previous expression is, by inspection, the same as that of the curly parentheses:

$$
g(\alpha;\{J\}) := \alpha(\prod_i C_i^2+\prod_i S_i^2)-2\prod_i(C_iS_i)
$$

One obtains:

- $\alpha = 0$ (zero mean spin glass) $\Rightarrow g(\alpha; \{J\}) < 0$;

- $\alpha = 1$ (ferromagnetic) $\Rightarrow g(\alpha; \{J\}) = (\prod_i C_i - \prod_i S_i)^2 > 0$;

- for all $J^{(l)}$, $g(\alpha; \{J\})$ is increasing function of $\alpha$;

- $\mathrm{Av}_{(K_h)}[K_h K_k(\omega_{hk} - \omega_h \omega_k)] = 0$ on the $(J^{(l)}, \alpha)$ plane curve with $J^{(l)} > 0$ and $0 \le \alpha \le 1$ defined by:

$$\alpha(J^{(l)}) = \frac{2 C_l S_l \prod_{i \neq l}(C_i S_i)}{C_l^2 \prod_{i \neq l} C_i^2 + S_l^2 \prod_{i \neq l} S_i^2} \tag{2.13}$$

The proof of the inequalities for one dimensional systems with free boundary conditions or for tree-like lattices is trivial since, due to the absence of loops the partition function factorizes

$$\mathbf{Z} = 2^N \prod_i \cosh(\lambda_i J_i)$$

and by consequence the first inequality is fulfilled even without taking the average and the second inequality reduces obviously to the equality to zero.

# 3 Comments

We proved that a one dimensional spin glass system fulfills a family of correlation inequalities without the assumption of translation invariance for the interaction distribution. The first inequality extends a similar one proved in [CL] for any lattice and any interaction with zero mean value. Here we have shown that the inequality is stable by suitable deformations of the zero mean hypotheses. The inequality of type II proved here shows that in the zero mean case the truncated correlation function has the opposite sign of the standard GKS inequality i.e. the case of interactions with zero variance and positive mean. We have moreover identifyed the line crossing which the truncated correlation changes its

sign. It would be interesting to establish if an inequality of type (1.2) is fulfilled also in higher dimensions (see [KNA]). In fact, as a straightforward computation shows in the Gaussian case, if such an inequality holds then the overlap expectation would be monotonic in the volume and several nice regularity properties would follow [CG2]. We also mention that the inequality (1.2) doesn't hold in general topologies as it was shown to us by Hal Tasaki for a Bernoulli spin chain with an extra bond connecting two non adjacent sites. Moreover a similar violation for the inequality (1.2) can be obtained in the case in which the disorder, still having zero average, is non symmetric.

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